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A Paradox of Definability: Richard's and Poincaré's Ways Out

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In 1905, Richard discovered his paradox of definability, and in a letter written that year he presented both the paradox and a solution to it. Soon afterwards, Poincaré endorsed a variant of Richard's solution. In this paper, I critically examine Richard's and Poincaré's ways out. I draw on an objection of Peano's, and argue that their stated solutions do not work. But I also claim that their writings suggest another way out, different from their stated solutions, and different from the orthodox Tarskian approach. I argue that this second solution does not prevent the return of the paradox.

1. Richard's paradox

Jules Richard and Julius König discovered the paradoxes that bear their names at almost exactly the same time, in the summer of 1905. König's paradox involved not only the notion of definability, but also the set-theoretical notions of ordinal number and well-ordering; König himself thought that the contradiction showed that the reals cannot be well-ordered.[1] But Richard's paradox made no mention of ordinal numbers and well-orderings. Richard's simple paradox revealed, in a non-technical way, a fundamental problem with the notion of definability.

Richard *1905* presents the paradox as follows. A certain set of numbers, the set E, is defined through the following considerations: write in alphabetical order all permutations of pairs of letters of the alphabet, followed by all permutations of triples of letters of the alphabet taken in alphabetical order, and so on for quadruples, quintuples etc. Cross out all permutations of letters that do not define numbers. E is the set whose members are u_1, the first number defined by a permutation, u_2, the second number defined by a permutation, and so on: that is, E is the set of all numbers that can be defined by finitely many words. Now, consider the following collection G of letters:

(G) Let p be the digit in the nth decimal place of the nth number of the set E; let us form a number having 0 for its integral part and, in its nth decimal place, $p+1$ if p is not 8 or 9, and 1 otherwise.

Call the number so defined N. Then N cannot belong to the set E. If it were the nth

1 Following König *1905*, consider those real numbers that can be defined by finitely many words of English. These reals form a denumerable set. Now consider the reals that cannot be so defined. If these reals can be well-ordered, then there is a least member. But now, as we have in effect just demonstrated, this 'undefinable' real *can* be defined in finitely many words. So we have a contradiction. König concluded that the reals cannot be well-ordered—but this conclusion is unacceptable in the light of Zermelo's well-ordering theorem (Zermelo *1904* and *1908*). So the contradiction remains, and we are confronted with a paradox of definability.

member in E, then the digit in its nth place would be the same as the digit in the nth decimal place of the nth number, which is not the case. Yet N is defined by finitely many words: hence it should belong to the set E. So we have a contradiction.

We should note in passing two easily corrected flaws in Richard's presentation. Though he refers to E as a set, he treats it as a sequence. And he overlooks the fact that different permutations of letters may define the same number.[2]

2. Richard's way out

Richard *1905* goes on to offer a solution to the paradox. I quote it in full:

> Let us show that this contradiction is only apparent. We come back to our permutations. The collection G of letters is one of these permutations; it will appear in my table. But, at the place it occupies, it has no meaning. It mentions the set E, which has not yet been defined. Hence I have to cross it out. The collection G has meaning only if the set E is totally defined, and this is not done except by infinitely many words. Therefore there is no contradiction.[3]

Richard does not make it clear why he thinks that the set E can only be totally defined by "infinitely many words". A natural reading is this: we can only totally define E by an infinite enumeration of the sequence of expressions which define numbers. And now we can take Richard's resolution in the following way. At the point at which we reach the collection of letters G, only a finite number of members of E will have been defined, and so E will not yet have been totally defined. So G will not have a meaning, and must be crossed out.

This reading is confirmed by Richard *1907*, where he reconstructs his solution as follows:

> It seemed to me easy enough to explain this paradox. Let G be the phrase that defines N. This phrase is an arrangement of words. Since the elements of E come from arrangements of words, in forming the set E we will encounter the phrase G. Suppose we encounter it at rank p. At this moment it does not have meaning, for at this moment the first p–1 elements of E are the only ones defined. Having no meaning, the phrase G must be crossed out.[4]

There is an immediate difficulty with Richard's way out. It is tempting to suppose that we might reinstate the contradiction by adding to the collection G the considerations through which he defines E. That is, we add to G the letters "where E is the set defined by the following considerations", followed by the letters of his definition of the set E. Then, if his considerations do totally define E, we have a collection of letters that define a number which ought to be in E, but isn't, and we

2 Here is one way to correct the second flaw: for any number that is defined by more than one permutation, we choose the permutation that is first in alphabetical order. Such a procedure is found in Poincaré: 'In order to classify the integers, or the points in space, I shall consider the sentence which defines each integer or each point. Since it can happen that the same number or the same point can be defined by many sentences, I shall arrange these sentences in alphabetical order and I shall choose the first among these' (*1909*, p. 48).

3 I have followed the translation in van Heijenoort *1967*, p. 143.

4 Richard *1907*, p. 95.

are landed in a contradiction again. This objection to Richard's way out was raised by Peano *1906*:

> But the class E is defined in the vocabulary of the common language. Therefore, if we substitute for E its definition, the result is that N is expressed by means of the vocabulary of the common language alone, and the antinomy remains.[5]

Richard *1907* responds to Peano's objection as follows:

> But then we can make this remark: the phrase G gives rise to a contradiction. Let p be its rank in the set E; if the phrase G defines a number N, let x be its nth digit. The phrase G says that the pth digit of N is equal to $\phi(x)$; so it says that
> $$\phi(x) = x.$$
> But by the definition of $\phi(x)$ we have $\phi(x) \gtrless x$. Then the phrase G says that the pth digit of the pth number in E is different from itself, which is absurd. So we must cross it out.[6]

With this response, Richard has shifted his ground. The question is: on what grounds do we cross G out? Richard's first reason turned on the position G occupies in the enumeration: since only finitely many members of E are defined by the time we reach G, G contains a term, viz. 'E', that is not totally defined. But now Richard provides a second reason. Richard appeals to the *diagonal nature* of the definition G.[7] We are to cross G out because if we assume that it does define a number, we obtain a contradiction.

3. Poincaré's way out

Poincaré *1906* endorsed Richard's way out.[8] But Poincaré provides his own twist to Richard's solution, and a third reason why we must cross G out. According to Poincaré, we cannot define the set E in terms of E itself. The members of E must be defined independently of E. Now consider N. N is a member of E since it is defined by G, a finite collection of letters. But this definition *does* rely on E. And so there is a *vicious circle* in our definition of E. We must avoid this vicious circle by restricting the set E to those numbers that can be defined by a finite number of words *that do not mention E*. According to Poincaré, G does not define a number because the definition of a member of E cannot rely on E, on pain of a vicious circle.[9]

Poincaré extends the diagnosis to other paradoxes, in particular that of Burali-Forti. Here we consider the set **Ord** of all ordinal numbers, and the ordinal number α that corresponds to the order type of this set. We are quickly led to a contradiction, for α is a larger ordinal number than any in the set of *all* ordinals, and so larger than itself. Again Poincaré points to a vicious circle, this time in the definition of **Ord**: one of the members of **Ord**, namely α, is defined in terms of the set **Ord** itself. We must, then, introduce not an unrestricted set of all ordinals, but

5 Peano *1906*, p. 351. Here, and elsewhere, I am indebted to Loredana Ziff for help with the translation of Peano's paper.
6 Richard *1907*, p. 95.
7 For more on diagonal arguments and diagonal definitions, see Simmons *1990*, and *1993*, Chapter 2.
8 Poincaré *1906*, p. 305 and p. 307.
9 See Poincaré *1906*, p. 307.

rather the set of all those ordinals that can be defined independently of the set itself (see Poincaré 1906, p. 307). An unrestricted definition of a set of *all* ordinals, like the unrestricted definition of Richard's set E of *all* numbers defined by finitely many words, contains a vicious circle. Such definitions Poincaré calls *non-predicative*. In general:

> . . . the definitions that must be regarded as non-predicative are those that contain a vicious circle.[10]

4. Peano: an objection and a response

Suppose that we accept Richard's way out, or Poincaré's variant. We agree that the collection G of letters does not define a number—because G is meaningless, since it mentions E and E is not yet defined, or because G is a contradiction-producing diagonal definition, or because the definition of E would otherwise contain a vicious circle. Then there is an immediate difficulty. For, since G does not define a number, we 'have to cross it out'. If we do cross out G, we *will* obtain an enumeration of sentences that define numbers.[11] And if E is the corresponding totally defined set of numbers, defined free of any vicious circle, then G acquires a meaning, and defines a number not in E. The problem for Richard is that, if we take his way out, G does not remain meaningless and no longer produces a contradiction. And the problem for Poincaré is that if we restrict the set E to the set of numbers defined by a finite number of words that do not mention E, then we *can* define a new number by a finite expression that mentions E, without any vicious circle; Poincaré's restriction itself provides the conditions for lifting that restriction. Either way, G emerges as a perfectly meaningful sentence that does define a number. In short, it is no solution to reject G as meaningless, for that very rejection will render G meaningful.

This difficulty is pointed out by Peano. Peano writes:

> If the phrase that defines N does not express a number, as was demonstrated above, then, when I calculate N, I pass by this phrase, which does not define a number, and the definition of N acquires a meaning. That is to say, if N does not exist, then it exists.[12]

10 Poincaré *1906*, p. 307; see also p. 316. According to Poincaré, a proof that is truly based on logical principles is composed of a series of propositions, where those serving as premises are identities or definitions, and the rest are deduced step by step from the premises. Proofs do not yield new truths—they appear to do so only because each step of the proof may not be made fully explicit. If we replace the various expressions that appear in a proof by their definitions, and trace back through the steps, all that remains will be identities, which reduce to 'a huge tautology' (p. 316). Poincaré concludes: 'La Logique reste donc sterile . . .' (*ibid.*)

Why, asks Poincaré, might we think otherwise? Suppose that some of the definitions are non-predicative, and contain a vicious circle. Then there will be no reduction to a tautology. If we fail to recognize the problematic nature of these definitions, we may be misled into thinking that a proof does yield new truths, that logic is *not* sterile. But if we admit non-predicative definitions, says Poincaré, then what we get are not new truths, but paradoxes: 'Dans ces conditions, la Logistique n'est plus sterile, elle engendre l'antinomie' (*ibid.*).

11 Notice that we are assuming here that G is the only problematic definition in our list. Since there will be many other problematic definitions, this is clearly an over-simplification. However, it is implicitly assumed by Richard, Poincaré, and Peano in their treatments of the paradox, and we shall accept it for the sake of our discussion.

12 Peano *1906*, p. 357.

Peano suggests that the problem arises from an ambiguity in the phrase that defines *N*. (Note that in what follows, Peano slips into ambiguity himself, using '*N*' to denote not only the number *N*, but also the phrase that defines the number *N*. We should take Peano's term 'the phrase *N*' to pick out the phrase *G*). Peano writes:

> The contradiction lies in the ambiguity of the phrase *N*. It is necessary to add in an explicit way, 'this phrase included' or 'this phrase excluded'.
>
> Then we cross out the ambiguous phrase *N*, and continue on. A little further on we find the phrases:
>
> N' = (phrase *N*), this phrase excluded
>
> N'' = (phrase *N*), this phrase included.
>
> N'' does not exist, for the reason given. N' represents a determinate number, belonging to the class *E*, and clearly different from all *other* members of *E*.[13]

According to Peano, the ambiguity in the phrase *G* that defines *N* derives from the indeterminateness of the 'common language'. Common languages—such as Italian, French or English—are not as definite or determinate as formal languages, such as the language of arithmetic. And the definition of *N* is not wholly contained within a formal, regimented language. Peano points out that the definition of *N* essentially contains terms of the common language—for example, the crucial phrase 'can be defined'. Peano splits the definition of *N* into three separate formulas. Two of these are purely mathematical.[14] But the indispensable third part is expressed in the common language.[15] The ineliminable presence of certain expressions of the common language gives rise to Richard's paradox. Peano writes:

> But the main weak point in Richard's remarkable example is this: the definition of *N* is given partly in symbols (Prop. 1,3, p. 150, p. 151), and partly not in symbols (Prop. 2). The non-symbolic part contains ideas of the 'common language', ideas very familiar to us but not determinate, and they cause all the ambiguity. Richard's example does not belong to mathematics but to linguistics; an element that is fundamental in the definition of *N* cannot be defined in an exact way (according to the rules of mathematics). From an element that is not well-defined, we can draw several mutually contradictory conclusions.[16]

We can sympathise in a broad way with Peano's diagnosis. The notion of definability is expressed in ordinary language—and it is this notion that renders the definition of *N* problematic. But Peano's disambiguation of *G* does not lead us out of the difficulty, as we shall now see. There is a preliminary question about the occurrences of the demonstrative expression 'this phrase' in Peano's N' and N''. Do

13 Peano *1906*, p. 357.

14 Peano presents the symbolic parts of the definition of *N* as formulas (1) and (3):

(1) $N = \Sigma[10^{-n} \text{ anti Cfr}_{-n}\, fn|n, \mathrm{N}_1]$

(3) $fn = \text{Valore min}_n[\mathrm{N}_1 \cap x \ni (\text{Valore } x \in \Theta)]$.

See Peano *1906*, p. 351 and p. 352.

15 Peano's formula (2) is:
'(2) Valore *n* = numero decimale, quem numero *n*, scripto in systema alphabetico, defini, secundo regulas de lingua commune' (Peano *1906*, p. 352.)

16 Peano *1957*, pp. 357–8.

they refer to the original ambiguous phrase, or to N' and N'' respectively? I shall assume the former, since if N' and N'' were directly self-referential phrases, then neither would define a number—but according to Peano, N' does define a number. Reworking Peano's terminology, then, we introduce the two phrases:

$$G' = \text{(phrase } G\text{)}, G \text{ excluded}$$

and

$$G'' = \text{(phrase } G\text{)}, G \text{ included.}$$

The phrase G itself makes implicit reference to the finite phrases of English that define numbers, since it mentions the set E.[17] According to Peano, we can read G in two ways, according to whether G is or is not one of these phrases. So, since G is ambiguous, we have to cross it out. But the phrases G' and G'' do not suffer this ambiguity.

Suppose we accept, following Peano, that G' defines a number, and that G'' does not. We cross out the problematic phrases G and G''. Once we have crossed out these problematic definitions, we may go on to obtain the finite English phrases that define numbers. Now G is not one of the unproblematic defining phrases to which G makes reference—any ambiguity in G is now removed. So G will define a number. Moreover, since G defines a number, so does G''. We have not escaped the original difficulty: in crossing out these definitions, we provide the very grounds for their reinstatement. And there is a further problem for Peano. He presumes that G' defines a number. Then G' is included among the finite phrases that define numbers. But this inclusion will lead to a contradiction, since G' is a diagonal definition. The problem with G' here is just like the one that led Peano to cross G'' out. To sum up: if, with Peano, we assume that G and G'' do not define numbers, we are led to the conclusion that they do; and if, with Peano, we assume G' does define a number, we are led to the conclusion that it does not.

5. A 'stage' solution

If we adopt the way out proposed by Richard and Poincaré, we 'cross out' the diagonal definition G, and continue. But this yields a denotation for G, and so G has to be reinstated as the definition of a number. Now we might take the reinstatement of G as one step in a process that repeatedly produces new numbers. Since G is a diagonal definition, it will define a new number, and we obtain an enlarged set of numbers definable by finite phrases. And by a further diagonal definition, we can in turn add to this set. The set of numbers defined by finitely many words will be constantly changing.

This process is incompatible with Richard's and Poincaré's stated solutions. The process is generated only if we take G to define a number—and according to Richard and Poincaré, G does not define a number. So it is remarkable that both Poincaré and Richard show some sensitivity to this process.

17 Notice that, taken literally, the addition of either 'this phrase excluded' or 'this phrase included' to G makes little sense. Consider, for example, the result of appending 'this phrase excluded' to G. No phrase is explicitly mentioned in G, and so the nature of the exclusion is not explicit. However, the phrase G mentions the set E, and this set is defined in terms of the finite phrases that define numbers. It is from this collection of phrases that G is to be excluded. Peano's addition refers to this collection in an elliptical way.

Consider first Poincaré. We have seen that Poincaré *1906* characterizes non-predicative definitions as those that contain a vicious circle. In a later work, Poincaré offers an *alternative* characterization of non-predicative definitions and classifications.[18] It goes like this:

> . . . we draw a distinction between two types of classifications applicable to the elements of infinite collections: the *predicative* classifications, which cannot be disordered by the introduction of new elements; the *non-predicative* classifications in which the introduction of new elements necessitates constant modification.[19]

Poincaré gives this illustration of a *predicative* classification:

> Let us suppose for example that we classify the integers into two families according to their size. We can recognize whether a number is greater or less than 10 without having to consider the relation of this number with the set of the other integers. Presumably, after the first 100 numbers have been defined, we shall know which among them are less than and which are greater than 10. When we then introduce the number 101, or any one of the numbers which follow, those among the first 100 integers which were less than 10 will remain less than 10, those which were greater will remain greater; the classification is predicative.[20]

Poincaré contrasts this with a *non-predicative* classification:

> In order to classify the integers, or the points in space, I shall consider the sentence which defines each integer or each point. Since it can happen that the same number or the same point can be defined by many sentences, I shall arrange these sentences in alphabetical order and I shall choose the first among these. With this as a condition, this sentence shall end with a vowel or with a consonant, and the classification can be made according to this criterion. But this classification would not be predicative; by the introduction of new integers, or of new points, sentences which had no meaning could acquire one. And then to the list of sentences which define an integer or a point already introduced, it will become necessary to add new sentences, which up to this point were devoid of meaning, have just acquired a meaning, and which define precisely this same point. It can happen that these new sentences assume the first position in the alphabetical order, and that they end with a vowel, whereas the previous sentences ended in a consonant. And then the integer or the point which had been provisionally placed in one category will have to be transferred to another.[21]

18 For Poincaré, a definition is a kind of classification. Poincaré writes: 'Every definition is, in effect, a classification. It separates the objects which satisfy the definition from those which do not, and it arranges them in two distinct classes. . . . A definition, like all classifications, may or may not be predicative' (Poincaré *1909*, p. 48).

19 Poincaré *1909*, p. 47.

20 Poincaré *1909*, p. 47.

21 Poincaré *1909*, p. 48.

The situation is similar with Richard's paradox. Here, we classify the numbers into two groups: those that are definable by finitely many words, and those that are not. Given our list of sentences that define numbers, we obtain the members of E. But once we have E, then there are sentences, like G, that were devoid of meaning and have just acquired a meaning. And now the number N, first placed in the category of numbers that were not definable in finitely many words, is placed in the other category. We obtain an expanded set of numbers that are definable by finitely many words of English. But then there will be sentences of English that make reference to this set (for example, this very sentence); and among these sentences there will be diagonal definitions that define a number not in this expanded set. Such sentences will lead to a further expansion of the set of numbers definable by finitely many words. So our classification requires constant modification. The classification is not immutable, and it is this, says Poincaré, that is the source of Richard's antinomy, and of others too. Poincaré writes:

> Formal logic is nothing but the study of the properties common to all classifications; it teaches us that two soldiers who are members of the same regiment belong by this very fact to the same brigade, and consequently to the same division; and the whole theory of the syllogism is reduced to this. What is, then, the condition necessary for the rules of this logic to be valid? It is that the classification which is adopted be *immutable*. We learn that two soldiers are members of the same regiment, and we want to conclude that they are members of the same brigade; we have the right to do this provided that during the time spent carrying on our reasoning one of the two men has not been transferred from one regiment to another.
>
> The antinomies which have been revealed all arise from forgetting this very simple condition: a classification was relied on which was not immutable and which could not be so ....[22]

Poincaré draws this moral: 'Avoid non-predicative classifications and definitions'.[23] In generating Richard's paradox, we rely on a non-predicative classification, between numbers that are finitely definable and those that are not. Following Poincaré, we might eschew this classification, and cross out the phrase G, since that phrase relies on the classification. Here we have a new reason to cross out G. *But still the old difficulty remains*. Our list will now be composed of the unproblematic definitions of numbers, those that rely on predicative classifications only. And now the classification between numbers that are finitely definable and those that are not is rendered predicative: we can place each number in one or the other category, according to whether or not it is defined by a phrase in our list. So we must reinstate G—and the paradox returns. We cannot take Poincaré's way out here.

In following Poincaré, we have regarded the procedure that constantly generates new numbers as a symptom of a non-predicative classification. But we need not regard it that way. We might instead take the procedure itself to embody a solution to the paradox. The idea is this: we turn the paradox into a demonstration that, given any set of numbers definable by an English sentence, we can always enlarge it. Paradox arises if we fail to keep separate the distinct stages of this procedure.

22 Poincaré *1909*, p. 45.
23 Poincaré *1909*, p. 63.

In the final paragraph of Richard *1905*, the initial stages of such a procedure are described. It follows directly after the one quoted above at the outset of section 2:

> We can make a further remark. The set containing [the elements of] the set E and the number N represents a new set. This new set is denumerably infinite. The number N can be inserted into the set E at a certain rank k if we increase by 1 the rank of each number of rank [equal to or] greater than k. Let us still denote by E the thus modified set. Then the collection of words G will define a number N' *distinct from* N, since the number N now occupies rank k and the digit in the kth decimal place or N' is not equal to the digit in the kth decimal place of the kth number of the set E.[24]

From these remarks, we can distinguish three stages. At the first stage, G is meaningless, and we cross it out; we obtain a set E of numbers definable by English phrases. At the second stage, with E totally defined from the first stage, G acquires a meaning and defines a number N, not in E. This leads to an enlarged set which Richard still calls E. At the third stage, since the term 'E' now refers to a new set, G refers to a new number N', and we are led to a further enlargement of E. This procedure continues indefinitely.

It is a peculiarity of Richard's presentation that the name 'E' refers to different sets of numbers. We can describe the procedure so as to avoid any such ambiguity. Let E be the set of numbers obtained at the first stage. Then, at the second stage, the diagonal definition G refers to a number not in E. Let E' be the set of numbers obtained at the second stage. There is a diagonal definition which is like G, except that the name 'E' is replaced by 'E''. This sentence acquires a meaning at the third stage, and defines a number not in E'. We can supply the name 'E''' for the set of numbers obtained at the third stage, and this will confer a meaning on a diagonal definition at the fourth stage; the procedure continues indefinitely.

We now have an unambiguous characterization of the procedure suggested by Richard. This 'stage' account suggests a solution to the paradox which is *different* from Richard's and Poincaré's original proposals. The idea is no longer that G must simply be crossed out (because it is meaningless *simpliciter* or a contradiction-producing diagonal definition, or because it contains a vicious circle or relies on a non-predicative classification). Now the idea is that we are to separate out stages in the procedure that apparently led to paradox. At each stage, a diagonal definition that had no meaning acquires one, and defines a number not previously defined. We do not reach a set of *all* numbers definable by an English phrase, but rather a sequence of increasingly comprehensive sets of numbers.

We should be careful to distinguish Richard's 'stage' solution from the orthodox Tarskian response to definability paradoxes. According to a simple Tarskian resolution of Richard's paradox, the predicate 'defines' is stratified into a series of distinct predicates: 'defines$_0$', 'defines$_1$', 'defines$_2$', ... The phrase 'the positive square root of four' defines$_0$ the number two, since the phrase does not contain the definition predicate. The phrase 'the square of the number defined$_0$ by "the positive square root of four"' defines$_1$ the number four, since the phrase contains the predicate 'defines$_0$'. And so on up the hierarchy. Each occurrence of the definition predicate is tied to a level. In particular, this holds for the occurrence of the

24 Richard *1905*, pp. 143–4.

definition predicate in G. If the level of this occurrence is n, then G is an unproblematic definition of level $n + 1$—and in this way paradox is avoided.

Richard is not proposing this kind of Tarskian stratification of the definition predicate. At the first level of the Tarskian hierarchy we find only defining phrases free of the definition predicate, like "the positive square root of four". But at the first stage of the procedure suggested by Richard, we have no reason to exclude defining phrases simply on the grounds that they contain the definition predicate. For example, the phrase 'the square of the number defined by "the positive square root of four"' contains the definition predicate, but this feature does *not* exclude it from the original list. Notice that definitions may be nested—consider the example in the previous sentence, prefixed by 'the number defined by the cube of'. Such nested definitions will take us higher in the Tarskian hierarchy—but nested definitions of arbitrarily high degree may appear in the original list. Even definitions that make reference to the list may appear at this first stage—consider the phrase 'the square root of the number defined by the preceding phrase'. For Richard, it is only the sentence G—a diagonal definition that refers to the set E—that must be excluded.[25]

6. Richard's paradox regained

Let us refer to the procedure we have just described as 'Richard's procedure'. Since the name 'Richard's procedure' has a determinate reference, its occurrence in a sentence of English cannot render that sentence meaningless. Notice also that Richard's procedure has *denumerably* many stages, since the sequence of names of English $E, E', E'', \ldots$ is denumerable. And since each of the denumerably many stages contains at most denumerably many sentences of English that define a number, Richard's procedure defines denumerably many numbers.[26] We can place these numbers in a definite order: the stages themselves are naturally ordered, and the English sentences in which they appear at any given stage can be grouped according to length and alphabetically ordered.

Now consider the following sentence H of English:

(H) Let p be the digit in the nth decimal place of the nth number defined by Richard's procedure; let us form a number having 0 for its integral part and, in its nth decimal place, $p + 1$ if p is not 8 or 9, and 1 otherwise.

The sentence H will occur in the enumeration of sentences generated at the first stage of Richard's procedure, and we will have to decide whether or not we should cross it out. There are two ways to proceed here. One way is to make it part of Richard's procedure that H—and any sentence that refers to the procedure itself—is crossed out at the first stage. We shall consider this way shortly.

The alternative is that we do not cross out H. After all, since the name 'Richard's procedure' has a determinate reference, the sentence H is itself meaningful. And

25 Again, this is an over-simplification (see note 11). G is not the only sentence that defines a number not in E, in terms of all the members of E. There will be other diagonal definitions, for example. But no phrase is excluded from the original listing merely because it contains the definition predicate.

26 Here I am utilizing a theorem of set theory, that a countable union of countable sets is countable.

though it *is* part of Richard's procedure to regard sentences at the first stage containing the names '*E*', '*E'*', '*E''*', . . . as meaningless, and to cross them out, it is *not* part of the procedure to regard sentences at the first stage which make reference to the procedure as meaningless.

Moreover, we should not cross out *H* simply on the grounds that it makes reference to an enumeration of which it is a member. Note that a similar circumstance obtains in recursion theory. It is well known that the diagonal argument raises a *prima facie* difficulty for any attempt to capture formally the notion of algorithmic function. Given a variety of formally characterized classes of algorithmic functions, the diagonal argument produces an algorithmic function falling outside the class.

The problem posed by the diagonal argument is resolved by allowing sets of instructions for non-total partial functions as well as for total functions. Suppose we have a list of sets of instructions for partial functions, ordered according to length and alphabetically. Let the $(x + 1)$st set of instructions be Q_x, and let ψ_x be the partial function determined by Q_x. Now define the partial function χ as follows: to compute $\chi(x)$, find Q_x, compute $\psi_x(x)$, and if and when a value for $\psi_x(x)$ is obtained, give $\psi_x(x) + 1$ as the value for $\chi(x)$. We have here instructions for computing the partial function χ, which we may suppose to be in our list. Suppose it is the $(x_o + 1)$st set of instructions; then $\chi = \psi_{x_o}$. But now

$$\psi_{x_o}(x_o) = \chi(x_o) = \psi_{x_o}(x_o) + 1$$

does not yield a contradiction because $\chi(x_o)$ may not have a value.

Notice that the set of instructions which determines χ refers to the entire list of sets of instructions; and further, this set of instructions is itself a member of the list. The set of instructions is not unintelligible or meaningless, and yet it refers to a list of which it is a member. Similarly, it would be a mistake to conclude that *H* is unintelligible or meaningless simply on the grounds that it refers to an enumeration of which it is a member.

Suppose, then, that we do not cross *H* out, but take it to define a number. Since *H* is a diagonal definition, it defines a number outside the set of numbers defined by Richard's procedure. Yet *H* is a sentence of English that is generated by Richard's procedure and that defines a number. We have a contradiction, and Richard's paradox re-emerges.

So perhaps we should avoid paradox by taking the other way: we revise the procedure so that the name 'Richard's procedure' is without a determinate reference at the first stage. At this first stage, we cross out any expression containing a reference to the procedure, just as we cross out expressions containing '*E*', or '*E'*', or . . . . But then, at what stage do we count the name 'Richard's procedure' as having a determinate reference? Presumably at *no* stage. For if there is some stage at which 'Richard's procedure' has a determinate reference, then the expression *H* is meaningful at that stage, and the paradox returns. For again, the number defined by *H* will be a member of the set of numbers defined by Richard's procedure, and will also fail to be a member of that set, since *H* is a diagonal definition.

But if the name 'Richard's procedure' has no determinate reference at any stage, then it must be that an English expression containing this name is never meaningful. More generally, we cannot allow any expression to pick out the procedure, for otherwise that expression may be used in a diagonal definition like *H*. The procedure cannot be meaningfully referred to. And then any attempt to solve the paradox via this procedure—including that of the last several paragraphs—cannot even be stated.[27]

Richard's attempt to resolve the paradox fails. His explicit resolution is to declare the diagonal definition G meaningless. Yet, if we take this way out, G is at once rendered meaningful. And so, as Richard himself seems to indicate, we are led to a 'stage' solution: we distinguish stages in a procedure that constantly produces new definitions and new numbers. But if we can refer to the procedure, we can produce a diagonal definition like H—and the paradox is regenerated. And if we cannot refer to the procedure, then it is ineffable and unsuited to serve as a solution to paradox.[28]

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27 This problem for the stage solution bears some structural similarity to a problem for the Tarskian resolution of Richard's paradox. A problem for the Tarskian account is this: no level of the hierarchy can accommodate statements about the entire hierarchy. Similarly, the problem here for the stage solution is that no stage can accommodate talk of the entire procedure. We should be careful, though, not to push the analogy too far—as we have seen, Richard is not offering a Tarskian solution.

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